Minimum Fuel Control of Linear Stochastic Systems with Applications to Midcourse Guidance

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The general problem of minimum fuel control of discrete time gaussian processes is examined. Necessary conditions are derived for the optimal control of systems with both process and measurement uncertainties. A fixed correction time strategy is studied and results are applied to a spacecraft midcourse guidance problem. A variable correction time strategy is also treated and it is shown that the fixed time solution is useful in determining the optimal variable correction time control. Numerical results are presented for an Earth-Mars mission employing both fixed and variable correction time strategies.

I. Introduction

AN important consideration in the design of many spacecraft guidance systems is the procedure for determining midcourse velocity corrections. A number of authors have examined various aspects of this problem and obtained many useful results. Battin[3] provided the first extensive investigation and proposed a technique for deciding when to apply corrections. Stern and Potter[11] dealt with the deterministic midcourse problem and obtained optimal correction times. Striebel and Breakwell[13] and Tung[14] utilized the calculus of variations to determine the minimum fuel linear controller subject to a constraint on terminal miss distance. Orford,[9] Pfeiffer[10] and Van Gelder et al.[16] all considered final value stochastic controllers with constraints placed on fuel expenditure. Finally Tung and Striebel[15] obtained the solution of a minimum fuel, fixed correction time problem.

In the work presented here, the general problem of minimum fuel control of discrete time linear stochastic systems is examined. A solution is obtained for the case in which both process noise and measurement errors are significant. This solution is applied to the fixed correction time midcourse spacecraft guidance problem, duplicating the results of Tung and Striebel.[15] A variable correction time problem is then postulated and solved with the help of the fixed correction time result. It is found that when the variable time strategy is used, regions in the space of estimated state vectors, determine the optimal corrections. If the estimated target miss vector lies in a predetermined region, the optimal control is zero. If, however, the estimated miss vector lies outside this region, the optimal correction drives the estimated state to the boundary of a second region contained within the first.

II. Statement of the General Problem

It is assumed that the plant may be modelled as a discrete time stochastic system. Transition of the state vector from time t_n to time t_{n+1} is described by the linear vector equation

$$x(n+1) = \Phi(n+1,n)\,x(n) + \theta(n+1,n)u(n) + v(n) \quad (1)$$

Vector $x(n)$ is the k dimensional state, $\Phi(n+1,n)$ is a $(k \times k)$ state transition matrix, $u(n)$ is a p dimensional vector of control variables, $\theta(n+1,n)$ is a $(k \times p)$ control gain matrix and $v(n)$ is a k dimensional vector of random process disturbances. The initial state $x(0)$ is a normally distributed vector valued random variable with known statistics

$$E[x(0)] = 0 \quad E[x(0)x^T(0)] = X(0) \quad (2)$$

Process disturbances $v(n)$ are independent gaussian vector valued random variables with statistics

$$E[v(n)] = 0 \quad E[v(n)v^T(i)] = \begin{cases} V(n) & i = n \\ 0 & i \neq n \end{cases} \quad (3)$$

Since the process disturbance vector $v(n)$ is independent of the state $x(n)$ and control $u(n)$, it is implicitly assumed that any control errors are statistically independent of the applied control and hence may be included in $v(n)$.

A feedback controller has the task of determining appropriate control vectors $u(n)$. The controller has vector valued measurements $m(n)$ available, which are determined by the linear equation

$$m(n) = H(n)\,x(n) + w(n) \quad (4)$$

Matrix $H(n)$ defines the available measurement and $w(n)$ is an independent gaussian random measurement error with statistics

$$E[w(n)] = 0, \quad E[w(n)w^T(i)] = \begin{cases} W(n) & i = n \\ 0 & i \neq n \end{cases} \quad (5)$$

Since the state and measurement vectors are subject to random disturbances, it is appropriate to consider a cost function which is the expectation of the total fuel plus a terminal penalty imposed on the final state. The cost is written as

$$J = E\left\{\sum_{n=1}^{q-1} \|u(n)\| + \phi[x(q)]\right\} \quad (6)$$

with the expectation in Eq. (6) implicitly conditioned on the a priori statistics of the system. Measurement and control both begin at time t_1 and end at time t_{q-1}. At each decision point t_n, control may be applied and the measurement history $m(1), m(2), \ldots, m(n)$ is available for determining the control. Time t_q is a specified terminal time and $\phi(x(q))$ is a scalar penalty function of the terminal state.

Using the definitions and assumptions explained above, the optimization problem can be stated in specific terms as follows: "Find the control $u(n)$, as a function of the past history of measurements up to time t_n, that will drive the state $x(n)$ so that the expected cost J is minimized." In what follows, the task of synthesizing this optimal control function will be examined in some detail and results applied to the problem of determining spacecraft midcourse velocity corrections.

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III. Optimal Control with Fixed Correction Time

It has been shown[6,12] that for the purpose of implementing the optimal control of a linear plant with additive gaussian disturbances, the measurement information available to the controller may be summarized in terms of the minimum variance state estimate $\hat{x}(n)$. Recursion formulas have been derived by Kalman[8] and Battin[3] for determining $\hat{x}(n)$ from measurements $m(n)$. For easy reference these formulas are repeated as follows:

$$\hat{x}(n) = \hat{x}'(n) + P'(n)H^T(n)[H(n)P'(n)H^T(n) + W(n)]^{-1}[m(n) - H(n)\hat{x}'(n)] \quad (7)$$

$$\hat{x}'(n+1) = \Phi(n+1,n)\hat{x}(n) + \theta(n+1,n)u(n) \quad \hat{x}(0) = 0 \quad (8)$$

$$P(n) = P'(n) - P'(n)H^T(n)[H(n)P'(n)H^T(n) + W(n)]^{-1}H(n)P'(n) \quad (9)$$

$$P'(n+1) = \Phi(n+1,n)P(n)\Phi^T(n+1,n) + V(n) \quad P(0) = X(0) \quad (10)$$

It is assumed that covariance matrices $P(n)$ may be computed a priori. All necessary information for determining the optimal control is embodied in $\hat{x}(n)$, so the controller may calculate $\hat{x}(n)$ recursively, using Eqs. (7–10) and implement the optimal control as a function of $\hat{x}(n)$. This procedure eliminates the need for an expanding memory controller and in cases for which many measurements are taken, greatly simplifies the task of synthesizing the optimal control function.

To realize the optimal control as a function of $\hat{x}(n)$, it is necessary to determine the minimum expected cost to complete the process from each control decision point. A recursion formula has been derived[6,12] for the minimum expected cost to complete the process conditioned on the measurement history. For the minimum fuel problem defined by Eqs. (1–6) this formula is

$$C^*[\hat{x}(n),n] = \min_{u(n)}\Big[\|u(n)\| + \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k f_{s(n+1)}(\xi) C^*[\hat{x}'(n+1) + \xi, n+1]\Big] \quad (11)$$

where $C^*(\hat{x}(n),n)$ is the minimum expected cost to complete the process from time t_n, given the measurement history up to t_n. Function $f_{s(n+1)}(\xi)$ appearing in Eq. (11) is a normal probability density

$$f_{s(n)}(\xi) = (2\pi)^{-k/2}S(n)^{-1/2}\exp[-\tfrac{1}{2}\xi^T S(n)^{-1}\xi] \quad (12)$$

with covariance matrix $S(n)$[6] given by

$$S(n) = P'(n)H^T(n)[H(n)P'(n)H^T(n) + W(n)]^{-1}H(n)P'(n) \quad (13)$$

Equation (12) represents the probability density of a random vector $s(n)$ identified as

$$s(n) = P'(n)H^T(n)[H(n)P'(n)H^T(n) + W(n)]^{-1} \times [m(n) - H(n)\hat{x}'(n)] \quad (14)$$

and from Eq. (7), it can be seen that $s(n)$ is the incremental change in the estimated state, as a result of processing the measurement $m(n)$. The extrapolated estimate $\hat{x}'(n+1)$, appearing on the right of Eq. (11), is determined from Eq. (8). Finally, it can be shown[6] that the proper terminal condition on Eq. (11) is the expected terminal penalty, conditioned on the estimated state.

$$C^*[\hat{x}(q),q] = E\{\phi[x(q)]|\hat{x}(q)\} = \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k f_{x(q)|\hat{x}(q)}(\xi)\phi(\xi) \quad (15)$$

where $f_{x(q)|\hat{x}(q)}(\xi)$ is the probability density of $x(q)$, conditioned on $\hat{x}(q)$

$$f_{x(q)|\hat{x}(q)}(\xi) = (2\pi)^{-k/2}|P_{(q)}|^{-1/2}x \exp\{-\tfrac{1}{2}[\xi - \hat{x}(q)]^T P(q)^{-1}[\xi - \hat{x}(q)]\} \quad (16)$$

It is important to note that C^* in Eq. (11) may be defined as a function of $\hat{x}(n)$ instead of the entire measurement history. This is possible because $\hat{x}(n)$ is a sufficient statistic[1,5] and embodies all the measurement information relevant to C^*. A backward step by step solution of Eqs. (11, 12, and 15) will determine the minimum cost to complete the process from all estimated states, at each decision point. This procedure is essentially the dynamic programming method of Bellman[4,7] with the added complication of evaluating the integral on the right of Eq. (11). Solution of Eqs. (11, 12, and 15) yields the optimal control as a function of $\hat{x}(n)$ and n

$$u^*(n) = \mu[\hat{x}(n),n] \quad (17)$$

To determine the optimal control, consider the minimum on the right of Eq. (11). Define a scalar valued function $C^{*\prime}(\zeta,n)$ as

$$C^{*\prime}(\zeta,n) = \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_k f_{s(n+1)}(\xi) C^*(\zeta + \xi, n+1) \quad (18)$$

and an extrapolated state estimate $\hat{y}(n)$ as

$$\hat{y}(n) = \Phi(n+1,n)\hat{x}(n) \quad (19)$$

Substituting Eq. (18) into Eq. (11) obtains

$$C^*[\hat{x}(n),n] = \min_{u(n)} \{\|u(n)\| + C^{*\prime}[\hat{x}'(n+1),n]\} \quad (20)$$

where, from Eq. (8) and Eq. (19)

$$\hat{x}'(n+1) = \Phi(n+1,n)\hat{x}(n) + \theta(n+1,n)u(n) = \hat{y}(n) + \theta(n+1,n)u(n) \quad (21)$$

In most cases of practical interest $f_{s(n+1)}(\xi)$ is a nonsingular probability density (i.e., $|S(n+1)| > 0$). If, in addition, the terminal penalty ϕ is a convex function of $x(q)$, then it can be shown[6] that $C^*(\zeta, n+1)$ is continuous in ζ. Under these conditions $C^{*\prime}(\zeta,n)$ is analytic in ζ and Eq. (18) may be expanded in Eq. (20) to obtain

$$C^*[\hat{x}(n),n] = \min_{u(n)} \left\{\|u(n)\| + C^{*\prime}[\hat{y}(n),n] + \left[\frac{\partial C^{*\prime}(\zeta,n)}{\partial \zeta}\right]_{\zeta=\hat{y}(n)} \theta(n+1,n)u(n) + r[u(n)]\right\} \quad (22)$$

where $\partial C^{*\prime}/\partial\zeta$ is the gradient (row vector) and the remainder term satisfies

$$|r[u(n)]| \leq \alpha\|u(n)\|^2 \quad 0 \leq \alpha < \infty \quad (23)$$

Equation (22) provides useful conditions for a minimum. If $\hat{y}(n)$ lies in a region $Z(n)$, of the k dimensional space of $\hat{y}$ vectors, such that

$$Z(n) = \left\{\hat{y}: \left\|\left[\frac{\partial C^{*\prime}(\zeta,n)}{\partial\zeta}\right]_{\zeta=\hat{y}} \theta(n+1,n)\right\| < 1\right\} \quad (24)$$

then $u(n) = 0$ yields a local minimum on the right of (22).† Now consider a region $R(n)$ given as

$$R(n) = \left\{\hat{y}: \left\|\left[\frac{\partial C^{*\prime}(\zeta,n)}{\partial\zeta}\right]_{\zeta=\hat{y}} \theta(n+1,n)\right\| > 1\right\} \quad (25)$$

This set is disjoint from $Z(n)$ and if $\hat{y}(n)$ lies in $R(n)$, inspec-

† This result follows from the fact that a neighborhood of $u(n)$ vectors, centered at 0, can always be chosen small enough so the remainder term in Eq. (22) is arbitrarily small.

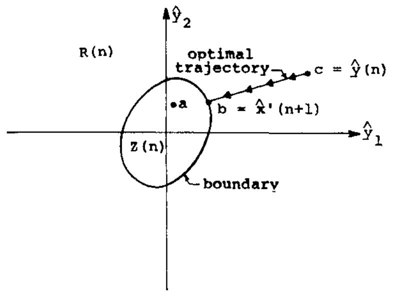

Fig. 1 Optimal trajectories in $\hat{y}$ space.

ion of Eq. (22) shows that a minimum cannot occur for $u(n) = 0$. To determine the minimum when $\hat{y}(n)\,\epsilon R(n)$, the derivative of the function on braces on the right of Eq. (20) is taken

$$\frac{\partial\{:\}}{\partial u(n)} = \frac{u^T(n)}{||u(n)||} + \left[\frac{\partial C^{*\prime}(\zeta,n)}{\partial \zeta}\right]_{\zeta=\hat{x}'(n+1)} \theta(n+1,n) \qquad \hat{y}(n)\epsilon R(n) \quad (26)$$

and a necessary condition for the minimum is

$$\frac{u^*(n)}{||u^*(n)||} = -\theta^T(n+1,n)\left[\frac{\partial C^{*\prime}(\zeta,n)}{\partial \zeta}\right]^T_{\zeta=\hat{x}'(n+1)} \quad \hat{y}(n)\epsilon R(n) \quad (27)$$

Condition (27) determines the direction of $u^*(n)$. If the absolute value of both sides is taken one obtains

$$\left|\left|\left[\frac{\partial C^{*\prime}(\zeta,n)}{\partial \zeta}\right]_{\zeta=\hat{x}'(n+1)} \theta(n+1,n)\right|\right| = 1 \qquad (28)$$

and comparing Eqs. (24, 25, and 28) it is clear that the optimal control must drive $\hat{x}'(n+1)$ to the boundary between $Z(n)$ and $R(n)$. Thus, to summarize, two necessary conditions for the optimal control are satisfied by the following control rules

$$u^*(n) = 0 \quad \text{if } \hat{y}(n)\epsilon Z(n) \qquad (29)$$

$$\frac{u^*(n)}{||u^*(n)||} = -\theta^T(n+1,n)\left[\frac{\partial C^{*\prime}(\zeta,n)}{\partial \zeta}\right]^T_{\zeta=\hat{x}(n+1)} \quad \text{if } \hat{y}(n)\epsilon R(n) \quad (30)$$

Figure 1 illustrates typical trajectories for a problem with two state variables. If $\hat{y}(n)$ lies at the point $a\epsilon Z(n)$, then the control is zero. If, however, $\hat{y}(n)$ lies at $c\epsilon R(n)$, then the control drives $\hat{x}'(n+1)$ to point b on the boundary. Point b and the corresponding control must satisfy Eqs. (21 and 30). It can be shown[6] that if the terminal penalty ϕ is a convex function of $x(q)$, then the conditions obtained above are also sufficient and the optimal control, as given by Eqs. (29) and (30), is unique.

IV. Midcourse Velocity Corrections at Fixed Times‡

Consider the midcourse phase of an interplanetary spacecraft mission. Because of random errors in the injection of of the spacecraft into its interplanetary trajectory, impulsive midcourse velocity corrections are necessary if the vehicle is to hit the target with sufficient accuracy. Chemical fuel rocket engines are used to perform these corrections. During the midcourse phase the spacecraft is tracked by ground based radars. These radars provide velocity measurements in the line of sight directions from the radars to the spacecraft, and the radar measurements are assumed to contain gaussian random errors. Estimates of spacecraft position and velocity are computed from the measurements by linearizing the equations of motion about a reference trajectory[3] and applying Eqs. (7–10). End points of the reference trajectory are the nominal injection and target points. The reference trajectory is assumed to lie in a plane and spacecraft deviations out of the plane are ignored. Except for injection errors there are no random disturbances to the spacecraft trajectory, so $V(n)$ in Eq. (10) is zero. Considering only in plane errors, deviations of the actual trajectory may be described by a four-dimensional (two coordinates of position and two coordinates of velocity) deviation state vector. Thus if a velocity correction is applied at time, t_n, the deviation at t_{n+1} is, to first order§

$$\delta(n+1) = \Phi(n+1,n)\delta(n) + \Phi(n+1,n)\begin{bmatrix}0\\I\end{bmatrix}\Delta v(n) \qquad (31)$$

where $\delta(n)$ = deviation state vector before correction (4-dimensional)

$\Phi(n+1,n)$ = state transition matrix (4×4), evaluated along the reference trajectory.

$\Delta v(n)$ = velocity correction vector (2-dimensional)

$\begin{bmatrix}0\\I\end{bmatrix}$ = compatability matrix (4×2)

Let time t_q be the nominal time of arrival at the target and define a linear transformation of state variables as follows:

$$\tilde{\delta}(n) = \Phi(q,n)\delta(n) \qquad (32)$$

so $\tilde{\delta}(n)$ is the deviation at time t_n extrapolated forward to the nominal time of arrival. Applying Eq. (32) to Eq. (31) yields the equation of state for $\tilde{\delta}(n)$

$$\tilde{\delta}(n+1) = \tilde{\delta}(n) + \Phi(q,n)\begin{bmatrix}0\\I\end{bmatrix}\Delta v(n) \qquad (33)$$

At the nominal time of arrival, there is a nonzero relative velocity V_R between a spacecraft on the reference trajectory and the target planet. Velocity V_R is assumed to lie in the plane of the reference trajectory. If the position components of $\tilde{\delta}(n)$ are resolved into a coordinate system such that an axis (1) lies in the trajectory plane orthogonal to V_R and axis (2) is parallel to V_R then position deviations at the nominal time of arrival are

$$\tilde{\delta}_1(n+1) = \tilde{\delta}_1(n) + \Phi_{13}(q,n)\Delta v_1(n) + \Phi_{14}(q,n)\Delta v_2(n) \qquad (34)$$

$$\tilde{\delta}_2(n+1) = \tilde{\delta}_2(n) + \Phi_{23}(q,n)\Delta v_1(n) + \Phi_{24}(q,n)\Delta v_2(n) \qquad (35)$$

where $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are position deviations orthogonal and parallel to V_R respectively.

Variable time of arrival guidance[3] is assumed, so no penalty is assigned to deviations $\tilde{\delta}_2$ parallel to V_R. Quadratic weighting is assigned to the miss distance at the target $\tilde{\delta}_1(q)$. Also, since the midcourse corrections are applied by chemical fuel rockets, the fuel required is proportional to the magnitudes of the corrections. Hence the cost function to be minimized becomes

$$J = E\left[\sum_{n=1}^{q-1} ||\Delta v(n)|| + \frac{\lambda}{2}\tilde{\delta}_1{}^2(q)\right] \qquad (36)$$

where λ is an arbitrary weighting factor. By making identifications

$$x(n) \to \tilde{\delta}_1(n) \qquad (37)$$

$$u(n) \to \begin{bmatrix}\Delta v_1(n)\\ \Delta v_2(n)\end{bmatrix} \qquad (38)$$

$$\theta(n+1,n) \to [\Phi_{13}(q,n)\Phi_{14}(q,n)] \qquad (39)$$

‡ This development follows closely the work of Tung and Striebel.[15]

§ It is tacitly assumed that there is small probability of large deviations, so that linearization provides an accurate model.

$$\Phi(n+1,n) \rightarrow 1 \tag{40}$$

$$\phi[x(q)] \rightarrow (\lambda/2)x^2(q) \tag{41}$$

it is apparent that Eqs. (34) and (36) are special cases of Eqs. (1) and (6). Further $\hat{x}(n) = \hat{\delta}(n)$ is the estimated miss distance at the target and from Eq. (21)

$$\hat{x}'(n+1) = \hat{y}(n) + \theta(n+1,n)u(n) = \hat{x}(n) + \Phi_{13}(q,n)\Delta v_1(n) + \Phi_{14}(q,n)\Delta v_2(n) \tag{42}$$

Applying the theory of Sect. III, it is clear that the optimal velocity corrections are functions of $\hat{x}(n)$, the estimated target miss distance. Furthermore, since $\hat{x}(n)$ is a scalar, $C^*(\hat{x}(n),n)$ is a function of one space variable so $\partial C^{*\prime}(\zeta,n)/\partial\zeta$ becomes a scalar. Thus if a nonzero velocity correction is applied, the optimal direction is determined from Eqs. (28 and 30) as

$$\frac{\Delta v^*(n)}{\|\Delta v^*(n)\|} = \pm \frac{\theta^T(n+1,n)}{\|\theta(n+1,n)\|} = \pm [\Phi^2_{13}(q,n) + \Phi_{14}(q,n)]^{-1/2} \begin{bmatrix} \Phi_{13}(q,n) \\ \Phi_{14}(q,n) \end{bmatrix} \tag{43}$$

Similarly, regions $R(n)$ and $Z(n)$ defined by Eqs. (24) and (25) become intervals on the real line and boundaries separating these intervals satisfy Eq. (28). For the problem at hand, it can be shown[15] that $\partial C^{*\prime}/\partial\zeta$ is a monotonic increasing antisymmetric function. Hence there are two boundary points of equal magnitude and opposite sign $\pm b(n)$ where

$$\left|\frac{\partial C^{*\prime}(\zeta,n)}{\partial\zeta}\right|_{\zeta=\pm b(n)} = \frac{1}{\|\theta(n+1,n)\|} \tag{44}$$

These points divide the $\hat{x}(n)$ axis up into intervals as shown in Fig. 2. If $\hat{x}(n)$ lies in either of the $R(n)$ intervals, the optimal control must drive $\hat{x}'(n+1)$ to the nearest of the boundary points $\pm b(n)$. If $\hat{x}(n)$ lies in $Z(n)$, then the optimal control is zero. Therefore, from Eqs. (29, 30, and 43) the optimal control becomes

$$\Delta v^*(n) = \begin{cases} 0 & \text{if } |\hat{x}(n)| \leq b(n) \\ \{\operatorname{sgn}[\hat{x}(n)]b(n) - \hat{x}(n)\} \dfrac{\theta^T(n+1,n)}{\|\theta(n+1,n)\|^2} & \text{if } |\hat{x}(n)| > b(n) \end{cases} \tag{45}$$

With knowledge of the boundary points $b(n)$, the optimal velocity correction strategy is completely determined. Points $b(n)$ may be computed a priori from the backward step by step solution of Eqs. (18, 28, and 20) with the terminal condition Eq. (15). For the particular problem considered here, these equations are written as follows:

$$C^*(\hat{x},q) = (\lambda/2)[\hat{x}^2(q) + P(q)] \tag{46}$$

$$C^{*\prime}(\zeta,n) = \int_{-\infty}^{\infty} d\xi \, \frac{\exp\{-\frac{1}{2}\xi^2/S(n+1)\}}{[2\pi S(n+1)]^{1/2}} \, C^*(\zeta+\xi,n+1) \tag{47}$$

$$\left[\frac{\partial C^{*\prime}(\zeta,n)}{\partial\zeta}\right]_{\zeta=b(n)} = \frac{1}{\|\theta(n+1,n)\|} \tag{48}$$

$$C^*(\hat{x},n) = \begin{cases} C^{*\prime}(\hat{x},n) & \text{if } |\hat{x}| \leq b(n) \\ \dfrac{|\hat{x}| - b(n)}{\|\theta(n+1,n)\|} + C^{*\prime}[b(n),n] & \text{if } |\hat{x}| > b(n) \end{cases} \tag{49}$$

where $S(n+1)$ is determined from Eqs. (13, 19, and 10) as

$$S(n+1) = P(n) - P(n+1) \tag{50}$$

and $P(n)$ is the error covariance of the estimated target miss distance after processing the measurement at time t_n. For

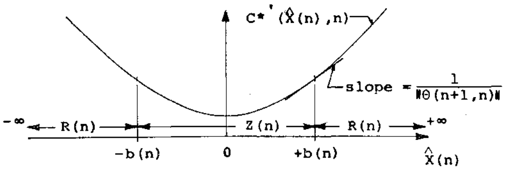

Fig. 2 Boundary points and regions $R(n)$, $Z(n)$.

the first step backward from time t_q, $C^{*\prime}$ is

$$C^{*\prime}(\zeta,q-1) = \int_{-\infty}^{\infty} d\xi \, \frac{\exp\{-\frac{1}{2}\xi^2/S(q)\}}{[2\pi S(q)]^{1/2}} \, \frac{\lambda}{2} [(\zeta+\xi)^2 + P(q)] = \frac{\lambda}{2} [\zeta^2 + S(q) + P(q)] = \frac{\lambda}{2} [\zeta^2 + P(q-1)] \tag{51}$$

and from Eq. (48)

$$b(q-1) = \frac{1}{\lambda\|\theta(q,q-1)\|} \tag{52}$$

with

$$C^*(\hat{x},q-1) = \begin{cases} \dfrac{\lambda}{2} [\hat{x}^2 + P(q-1)] & \text{if } |\hat{x}| \leq b(q-1) \\ \dfrac{|\hat{x}| - b(q-1)}{\|\theta(q,q-1)\|} + \dfrac{\lambda}{2} [b^2(q-1) + P(q-1)] & \text{if } |\hat{x}| > b(q-1) \end{cases} \tag{53}$$

Since $C^*(\hat{x},q-1)$ is not a quadratic function of $\hat{x}$, as was $C^*(\hat{x},q)$, the backward step to $q-2$ must be performed by approximation on a digital computer. The infinite integral Eq. (47) is approximated by considering a finite interval of interest on the ξ axis and applying linear extrapolation outside the interval.¶ Boundary points are determined from a search of the ζ axis for the value satisfying Eq. (48). Then C^* is obtained from Eq. (49) and the entire process repeated until the first correction time is reached. An actual numerical example is presented in Section VI.

V. Midcourse Velocity Corrections at Arbitrary Times

In the previous section, an optimal midcourse velocity correction strategy was determined for a problem with fixed correction times. In many practical situations it is not necessary to assign correction times a priori, but rather the controller may be allowed to choose the time of correction based on the measurement data. Commonly, however, it is necessary to restrict the maximum number of corrections allowed due to a limited engine restart capability or because of other operational limitations. In what follows, the optimal corrections and times to correct will be determined with a constraint placed on the maximum number of possible corrections. This solution makes use of the fixed correction time result described in Section IV.

The system model used in Section IV is applicable. Equation (36) defines the cost function to be minimized and index n denotes times t_n when velocity corrections may be applied. At each decision point, the controller must determine whether to apply a correction and if so, the direction and magnitude of the optimal correction. The maximum number of corrections r is specified a priori, and in general $r \ll q-1$. Hence

¶ It can be shown[6] that $f_{s(n+1)(\xi)}$ is the Green's function for a heat diffusion equation. Hence Eq. (47) may be obtained by applying central difference methods to realize the solution of the appropriate partial differential equation.

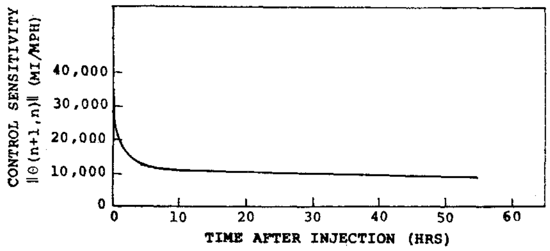

Fig. 3 Control sensitivity.

there are usually a large number of possible correction times but only a few corrections are allowed. As before, Eq. (34) describes the spacecraft state and with the identifications (37–41), the extrapolated state estimate satisfies Eq. (42).

As a means of keeping track of the number of velocity corrections which have been applied, an auxiliary state variable $a(n)$ is defined as

$$a(n+1) = a(n) + \text{sgn}[\|\Delta v(n)\|] \quad a(0) = 0 \tag{54}$$

where sgn(.) is identified as

$$\text{sgn}(\zeta) = \begin{cases} +1 & \text{if } \zeta > 0 \\ 0 & \text{if } \zeta = 0 \end{cases} \tag{55}$$

Thus $a(n)$ denotes the number of nonzero velocity corrections that were applied before time t_n. Since no more than r corrections are allowed, the following restriction is placed on the control.

$$\Delta v(n) = 0 \quad \text{if } a(n) = r \tag{56}$$

Applying the theory of Section III obtains a recursion formula for the expected cost to complete the process

$$C^*[a(n),\hat{x}(n),n] = \min_{\Delta v(n)} \Big[\|\Delta v(n)\| + \int_{-\infty}^{\infty} d\xi\, f_{s(n+1)}(\xi) \times C^*[a(n+1),\hat{x}'(n+1) + \xi, n+1]\Big] \tag{57}$$

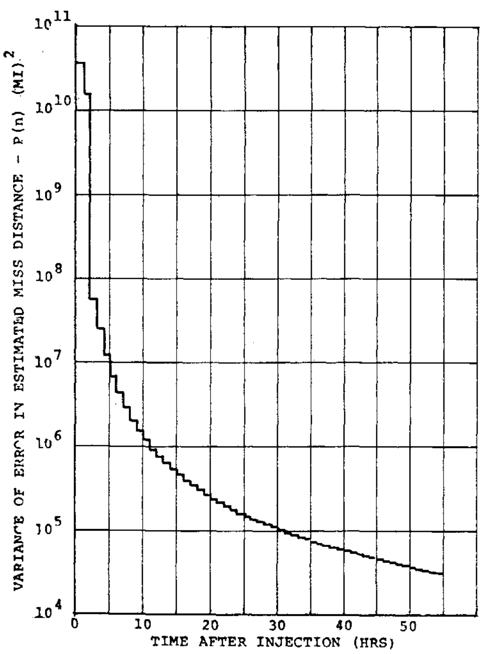

Fig. 4 Estimation error variance.

Note that C^* is a function of both the estimated state variable $\hat{x}(n)$ and the known state variable $a(n)$. Further, if $a(n) = r$, the control constraint Eq. (56) does not allow a correction, so minimization in Eq (57) cannot be carried out. Defining the auxiliary function $C^{*\prime}$ as

$$C^{*\prime}(m,\zeta,n) = \int_{-\infty}^{\infty} d\xi\, f_{s(n+1)}(\xi) C^*(m,\zeta + \xi, n+1) \tag{58}$$

Eq. (57) may be rewritten as

$$C^*[a(n),\hat{x}(n),n] = \min_{\Delta v(n)} \{\|\Delta v(n)\| + C^{*\prime}[a(n+1),\hat{x}'(n+1),n]\} \tag{59}$$

where $a(n+1)$ and $\hat{x}'(n+1)$ are given by Eqs. (42) and (54).

Now assume, for the time being, that the controller chooses to make a nonzero velocity correction at time t_n so from Eq. (54)

$$a(n+1) = a(n) + 1 \tag{60}$$

For any given value of $a(n+1)$, Eq. (59) is equivalent to Eq. (20) and the results of Section IV may be applied. In particular, from Eq. (45), the best nonzero correction is

$$0^+ \quad \text{if } |x(n)| \leq b[a(n),n] \tag{61}$$

$$\{\text{sgn}[\hat{x}(n)b[a(n),n] - \hat{x}(n)\} \frac{\theta^T(n+1,n)}{\|\theta(n+1,n)\|} \quad \text{if } |\hat{x}(n)| > b[a(n),n] \tag{62}$$

where 0^+ is a correction of infinitesimal magnitude which serves only to increment $a(n+1)$, as assumed at the onset. Note that the boundary point b is a function of both $a(n)$ and n satisfying

$$\left[\frac{\partial C^{*\prime}[a(n)+1,\zeta,n]}{\partial \zeta}\right]_{\zeta = \pm b(a(n),n)} = \frac{1}{\|\theta(n+1),n\|} \tag{63}$$

Hence the expected cost to complete the process using control Eqs. (61) and (62) is

$$D^+[a(n),\hat{x}(n),n] = \begin{cases} C^{*\prime}[a(n)+1,\hat{x}(n),n] & \text{if } |\hat{x}(n)| \leq b[a(n),n] \\ \dfrac{|\hat{x}(n)| - b[a(n),n]}{\|\theta(n+1,n)\|} + C^{*\prime}\{a(n)+1,b[a(n),n],n\} & \text{if } |\hat{x}(n)| > b[a(n),n] \end{cases} \tag{64}$$

If, however, the controller decides not to apply a correction, then the expected cost to complete the process is

$$D^0[a(n),\hat{x}(n),n] = C^{*\prime}[a(n),\hat{x}(n),n] \tag{65}$$

Therefore the choice of control is narrowed to two possibilities, zero control with expected cost D^0 and the best nonzero control with expected cost D^+. It follows that Eq. (59) may be written as

$$C^*[a(n),\hat{x}(n),n] = \text{MIN}\{D^0[a(n),\hat{x}(n),n];\; D^+[a(n),\hat{x}(n),n]\} \tag{66}$$

where MIN operation signifies choice of the smaller argument. For values of $\hat{x}(n)$ and $a(n)$ for which

$$D^0[a(n),\hat{x}(n),n] \leq D^+[a(n),\hat{x}(n),n] \tag{67}$$

the optimal control is zero and for

$$D^0[a(n),\hat{x}(n),n] > D^+[a(n),\hat{x}(n),n] \tag{68}$$

Table 1 Matched conic orbital elements

Trajectory	Semimajor axis (miles)	Eccentricity
Geocentric hyperbola	31,300	1.103
Heliocentric ellipse	117.3×10^6	0.208

the optimal control is given by Eq. (61) or Eq. (62). Thus for a given value of $a(n)$, values of $\hat{x}(n)$ are partitioned into regions of zero and non-zero control, according to Eqs. (67 and 68). Boundaries of these regions are determined by equality in Eq. (67). For most problems of practical interest, only two such boundaries are obtained and they are placed symmetrically about the origin. Hence if $\pm d(a(n),n)$ are the boundary points, they satisfy

$$\{D^0[a(n),\zeta,n] = D^+[a(n),\zeta,n]\}_{\zeta=\pm d(a(n),n)} \tag{69}$$

and the optimal control becomes

$$\Delta v^*(n) = \begin{cases} 0 & \text{if } |\hat{x}(b)| \leq d[a(n),n] \\ \{\text{sgn}[\hat{x}(n)]b[a(n),n] - \hat{x}(n)\} \dfrac{\theta(n+1,n)}{\|\theta(n+1,n)\|} & \text{if } |\hat{x}(n)| > d[a(n),n] \end{cases} \tag{70}$$

Note that the 0^+ control of Eq. (61) does not occur because in general $d[a(n),n] > b[a(n),n]$. This result is intuitively correct since the 0^+ control serves only to increment $a(n + 1)$ and its sole effect is to waste one of the available corrections. Mathematically it can be shown that $C^{*\prime}[a(n), \hat{x}(n),n] > C^{*\prime}[a(n) + 1,\hat{x}(n),n]$ so examination of Eq. (64) and Eq. (65) finds that condition Eq. (68) cannot occur for $|\hat{x}(n)| \leq b[a(n),n]$.

VI. A Numerical Example

A hypothetical Earth-Mars spacecraft mission is used to demonstrate the actual solution of a numerical example. Many body gravitational effects do not appreciably influence the minimization problem so matched conics are utilized to generate the reference trajectory. A geocentric hyperbola is used inside the Earth sphere of influence and a heliocentric ellipse connects the matching point on the Earth sphere of influence to a target point on the Mars sphere of influence. The geocentric hyperbola begins at injection, 100 miles above the Earth, and matches the heliocentric ellipse in position and velocity at a point 425,400 miles from the center of the Earth. Transfer angle of the geocentric hyperbola is 135°. The heliocentric ellipse is a Hohmann transfer. Orbital elements are given in Table 1. Total time on the reference trajectory is 6160 hr, of which the first 55 hr are spent on the geocentric hyperbola.

The guidance scheme uses both fixed and variable correction time methods. Two corrections are allowed. The first correction may be applied at any one of the integer hours after injection (1,2,3, . . . , 53 or 54 hr). The second correction must be applied at the Earth sphere of influence, 55 hr after injection. The matched conic reference trajectory is used to obtain state transition matrices at each integer hour and Eq. (39) determines the control sensitivity vectors $\theta^T(n + 1,n)$. It was found that for the first 55 hr of flight, these vectors are very nearly parallel to the reference trajectory velocities. Thus, according to Eq. (43) the optimal corrections are always applied tangent to the reference trajectory. Magnitudes of the control sensitivity vectors are plotted in Fig. 3.

The spacecraft is assumed to be an unmanned probe and variances for injection errors are chosen typically for such a mission, Table 2. Within the Earth sphere of influence, measurements of velocity are taken every hour by ground based radar. Error variances for these measurements are

Table 2 Injection error variances

	Altitude	Range
Position variance	1 (mile)^2	16 (mile)^2
Velocity variance	400 (mph)^2	20 (mph)^2

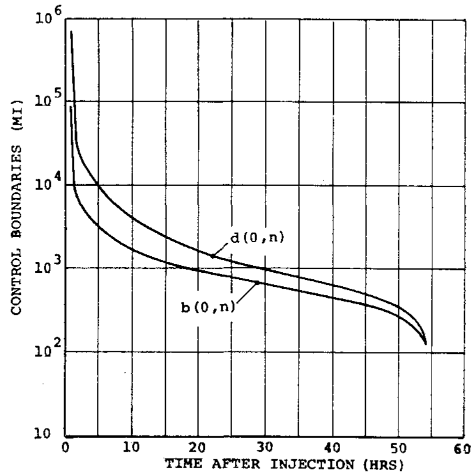

Fig. 5 Control boundaries.

0.01 (mph)^2. With these statistics the variances of errors in estimated target miss distance are determined at each measurement, Fig. 4.

High accuracy is required at the target so a relatively heavy terminal penalty is imposed on the target miss distance [λ = 0.0002 (mph)/(mile)^2]^2. This value corresponds to a one mile per hour penalty for a 100 mile miss distance at the target. Applying λ and $\theta(n + 1,n)$ at 55 hr to Eq. (52) obtains the threshold for the final correction [$b(55)$ = 0.512 mile]. Thus, the threshold is very small, relative to the other parameters of the problem and the final correction is nearly a total correction, essentially nulling the estimated miss distance at the target.

Since only one correction is allowed before 55 hr, the state variable $a(n)$ can only have values 0 and 1. Applying Eqs. (58, 63, 64, 65, and 66) and solving them by approximation on a digital computer obtains values for the boundary points $b(0,n)$ and $d(0,n)$ as shown in Fig. 5. These values were obtained by Balsamo and Edwards,[2] who developed efficient numerical methods for accurate computation of the boundary points. The first correction is applied whenever the estimated miss distance magnitude exceeds $d(0,n)$ and the optimal correction drives $\hat{x}'(n + 1)$ to $\pm b(0,n)$ (whichever is nearer). The optimal control law is Eq. (70). It was found that the expected cost for the two-correction strategy amounted to 14.1 mph expected fuel penalty, as compared with 22.1 mph. expected fuel when only a single-optimal correction is allowed at 55 hr.

Referring to Fig. 5, it is possible to find some intuitive justification for the numerical results. It can be seen that after the measurement at two hrs; there is a large reduction in threshold values. Similarly, from Fig. 4, the measurement at two hours produces a sizable reduction in estimation error covariance and the control sensitivity remains relatively high (Fig. 3). Thus two hours after injection is a favorable time and the narrowing thresholds, at that point, produce a high probability of correction. After two hours the thresholds decrease gradually indicating the effect of slowly decreasing estimation error variance.

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Engineering Estimates for Supersonic Flutter of Curved Shell Segments

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Theoretical flutter boundaries are given for cylindrical shell segments. The problem was motivated by portions of the Saturn V booster. Donnell's cylinder equations are used in conjunction with Galerkin's method. Two static aerodynamic theories are used, one based on Ackeret theory and the other based on slender body theory. These represent short and long wavelength theories, respectively. In addition a parameter has been included that typifies the spatial pressure distribution. As this parameter is varied in a continuous manner, one can observe the effect on the panel stability of passing from the short to the long wavelength theory. The result is an upper and lower estimate for the flutter boundary, yielding a thickness requirement as a function of panel curvature and length-to-width ratio. The segments are not sensitive to the spatial pressure distribution in the short wavelength region. This may account for the relative success of Ackeret theory in predicting cylinder flutter to date.

Nomenclature

D	= $Eh^3/[12(1-\nu^2)]$
F	= Airy stress function
H	= thickness parameter, $\{[M-1]^{1/2}E/[(1-\nu^2)q]\}^{1/3}h/L$
$\tilde{H}$	= thickness parameter, $[E/(1-\nu^2)q]^{1/3}\,h/L$
h	= panel thickness
L	= length of panel
M	= Mach number
m	= axial wave number
N	= number of modes
N_x,N_θ	= stress resultants; see Eqs. (5) and (6)
$p(x,\theta,t)$	= aerodynamic load
q	= integer, also dynamic pressure
R	= radius
t	= time
V	= flow velocity
W	= width of panel
W_{eff}	= effective width of panel, W/n
w	= panel displacement in radial direction
x	= spatial coordinate, flow direction
Z	= curvature parameter, $(L/R)\,(L/h)\,(1-\nu^2)^{1/2}$
δ_{qm}	= Kronecker delta
θ	= angular coordinate
θ_0	= included angle of shell segment
λ	= eigenvalue
ρ	= fluid density
ρ_s	= panel density
Ψ	= spatial phase shift
ω	= frequency, rad/sec

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I. Introduction

THIS theoretical study concerns the aeroelastic instability of a cylindrical shell segment. The problem was motivated by the need for design criteria for portions of the external structure of the Saturn V booster. The panel is a rectangular plate bent to a cylindrical shape and is freely supported on all four sides (Fig. 1). One might imagine the segment to be surrounded by rigid structure (Fig. 2), although the present analysis does not take the detail of the surrounding structure into account. Supersonic flow is directed parallel to the generators of the cylindrical shell segment. Primary interest is in low aspect ratio panels.

The structural side of the problem has been studied in a conventional way, with the use of Donnell's cylinder equa-